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Global bifurcation of solutions to diffusive logistic equations on bounded domains subject to nonlinear boundary conditions

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We consider the diffusive logistic equation

$$\frac{\partial u}{\partial t} = d\nabla^2 u + ru(1-u),$$

supplemented by the nonlinear boundary condition

$$\alpha(u)\nabla u \cdot \boldsymbol{\eta} + (1-\alpha(u))u = 0,$$

where α is a non-negative, non-decreasing function with $\alpha([0,1]) \subseteq [0,1]$. When regarded as an ecological model for an organism inhabiting a focal patch of its habitat, the assumptions on α are intended to capture a tendency on the part of the organism to remain in the habitat patch when it encounters the patch boundary that increases with species density. Such a mechanism has been suggested in the ecological literature as a means by which the dynamics of the organism at the scale of the patch might differ from its local dynamics within the patch. Building upon earlier examinations of the boundary-value problem by Cantrell and Cosner, we detail in this paper the global disposition of biologically relevant equilibria when both 0 and 1 (the local carrying capacity within the patch) are equilibria. Our analysis relies on global bifurcation theory and estimates for elliptic and parabolic partial differential equations.

1. Introduction

In this paper we continue the examination of the diffusive logistic model

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= d\nabla^2 u + ru(1-u) \quad \text{in } \Omega \times (0,\infty), \\ \alpha(u)\nabla u \cdot \boldsymbol{\eta} + (1-\alpha(u))u &= 0 \quad \text{on } \partial\Omega \times (0,\infty), \end{aligned}\right\} \tag{1.1}$$

which began in [5,6]. In (1.1), $u = u(x,t)$ denotes the density of a biological species at spatial location x and time t, and Ω designates a focal patch of habitat for the species. Mathematically, Ω is a bounded open domain in $\mathbb{R}^N$ (in applications N is usually 1, 2 or 3) with sufficiently smooth boundary, d and r are positive parameters giving the diffusion rate and intrinsic growth rate for the species, respectively, and the carrying capacity in (1.1) has been scaled to 1. The novel aspect of (1.1) lies in the inclusion of a density-dependent coefficient in the boundary condition. Throughout [5,6], $\alpha(u)$ is assumed to be smooth, non-negative and non-decreasing and to satisfy

$$\alpha([0,1]) \subseteq [0,1]. \tag{1.2}$$

In this modelling context, a Robin boundary condition of the form

$$\alpha^* \nabla u \cdot \boldsymbol{\eta} + (1-\alpha^*)u = 0, \tag{1.3}$$

where α^* is a fixed value in $[0,1]$, reflects in general terms the tension between the tendency of the organism to leave the focal patch Ω when it reaches the boundary and its tendency to return into the patch upon reaching the boundary. The tendency to remain in Ω becomes more pronounced as α^* increases, while the tendency to leave Ω upon reaching the boundary becomes greater as α^* decreases. When $\alpha^* = 1$, all organisms return into Ω upon reaching the boundary (a homogeneous Neumann or reflecting boundary condition), while, at the other extreme, when $\alpha^* = 0$, all organisms leave Ω upon reaching the boundary (a homogeneous Dirichlet or absorbing boundary condition). In this light, the assumptions on $\alpha(u)$ in [5,6] may be interpreted as saying that the propensity of the organism in question to remain in the patch upon reaching the boundary depends on the density of conspecifics along the boundary. The organism is more likely to remain in the patch if the density of conspecifics is relatively high along the boundary and less likely to remain if the density there is low. Our motivation for incorporating such a feature into the model was some empirical work on the Glanville fritillary butterfly [8].

In [8], the kind of density-dependent emigration from habitat patch boundaries that we assume in [5,6] is shown to be a possible mechanism for inducing an Allee effect [1,6] in the dynamics of the Glanville fritillary butterfly. An Allee effect is a widely studied concept in ecology. Very roughly, it refers to a situation in which the per capita population growth rate for a species declines with declining density when the density is below a threshold level. Initially, our primary aim in formulating (1.1) was to explore in a spatially explicit analytic modelling context whether one might induce an Allee effect in the dynamics of an organism at the scale of a habitat patch via density-dependent emigration from the patch along the boundary, and we provide an affirmative answer to the question in [6]. For more detail on the biological ramifications of the results and a discussion of Allee effects, we refer the reader to [5,6].

As indicated, the initial purpose in [6] was to establish a modelling context in which we could demonstrate that density-dependent emigration of an organism along a habitat patch boundary could induce an Allee effect in the dynamics of the organism in the habitat patch itself. Indeed, it is well known [4] that positive solutions to a diffusive logistic equation in a bounded domain, when subject to a fixed boundary condition of the form (1.3), either tend over time to a unique positive equilibrium or tend over time to the zero equilibrium. Neither of these outcomes constitutes an Allee effect. Hence, any Allee effect in the dynamics of an organism modelled by (1.1) is attributable to the density-dependent term $\alpha(u)$. In examining

(1.1), we found that we could readily employ either a principle of linearized stability or the method of upper and lower solutions to detect an Allee effect under suitable conditions on $\alpha(u)$. However, we also quickly realized that the inclusion of $\alpha(u)$ substantially complicated the structure of the solutions to the problem, in particular the structure of the positive equilibria. This realization has led to an exploration of the mathematical structure of the positive solutions of (1.1) beyond that which was needed to address the original biologically motivated question. Our first results in this vein appeared in [5] and the current paper addresses some questions left unanswered in that work.

In [5], we assume that α is non-negative, smooth and non-decreasing, and that it satisfies (1.2). Beyond these basic assumptions, the most significant factors in understanding the structure of positive equilibria to (1.1) are the values assigned to $\alpha(0)$, $\alpha(1)$ and $\alpha'(1)$. Equilibria to (1.1) are solutions to

$$\left.\begin{aligned} \nabla^2 u + \lambda u(1-u) = 0 \quad &\text{in } \Omega, \\ \alpha(u)\nabla u \cdot \boldsymbol{\eta} + (1-\alpha(u))u = 0 \quad &\text{on } \partial\Omega, \end{aligned}\right\} \tag{1.4}$$

where $\lambda = r/d \geqslant 0$. Whatever the value of $\alpha(0) \in [0,1]$, there is a continuum of positive solutions to (1.4) emanating from the ray $\{\lambda : \lambda \geqslant 0\} \times \{0\}$ of trivial equilibria to (1.1) in $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$ at the point $(\lambda^1_{\alpha(0)}(\Omega), 0)$, where $\lambda = \lambda^1_{\alpha(0)}(\Omega)$ is the unique non-negative value for which

$$\left.\begin{aligned} \nabla^2 \phi + \lambda\phi = 0 \quad &\text{in } \Omega, \\ \alpha(0)\nabla \phi \cdot \boldsymbol{\eta} + (1-\alpha(0))\phi = 0 \quad &\text{on } \partial\Omega \end{aligned}\right\} \tag{1.5}$$

admits a positive solution. This continuum necessarily satisfies the alternatives of the celebrated global bifurcation theorem of Rabinowitz [4, 9]. When $\alpha(0) = 0$, the boundary condition in (1.5) reduces to a homogeneous Dirichlet condition. Note that if $\alpha(0) = 0$, solutions to the diffusive logistic equation vanishing on $\partial\Omega$ are always solutions to (1.4). Note also that in this case the boundary condition in (1.4) factors into

$$u(\beta(u)\nabla u \cdot \boldsymbol{\eta} + (1-\alpha(u))) = 0, \tag{1.6}$$

where $\beta(u)$ is $\alpha(u)/u$ for $u \neq 0$ and $\beta(0) = \alpha'(0) \geqslant 0$. It follows that (1.4) can be solved in more than one way, so that, strictly speaking, (1.4) is not well posed in this case. However, if

$$\alpha'(0) > 0, \tag{1.7}$$

we may assume that $\beta(\mathbb{R}) \subseteq (\delta, R)$ for some fixed $R > \delta > 0$. In this case, if the boundary condition in (1.1) is replaced by

$$\beta(u)\nabla u \cdot \eta + (1-\alpha(u)) = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{1.8}$$

the problem is well posed by [3] and moreover, solutions to (1.1) corresponding to positive initial data throughout $\bar{\Omega}$ are uniquely determined as solutions to the diffusive logistic equation satisfying (1.8). We assume in [5] that (1.7) holds if $\alpha(0) = 0$. In this case, we show in [5] that if Ω is not a ball in $\mathbb{R}^N$, the continuum of positive equilibria to (1.1) emanating in $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$ from the ray of trivial solutions at the point $(\lambda^1_0(\Omega), 0)$ (such solutions necessarily satisfy homogeneous

Dirichlet boundary conditions) is isolated from positive equilibria to (1.1) which satisfy (1.8). (In the case in which Ω is a ball, these different types of positive equilibria to (1.1) can sometimes link up.)

If $\alpha(1) = 1$, then points along the ray $\{(\lambda, 1) : \lambda \in \mathbb{R}\}$ in $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$ are also positive equilibrium solutions to (1.1). We show in [5] that equilibria to (1.1) whose values in Ω lie in the open interval $(0, 1)$ bifurcate from this ray of equilibria at the point $(\lambda_0, 1)$, where $\lambda_0 \geqslant 0$ is the unique point at which the eigenvalue problem

$$\left.\begin{aligned} \nabla^2 \phi + \lambda \phi = 0 \quad &\text{in } \Omega, \\ \nabla \phi \cdot \boldsymbol{\eta} - \alpha'(1)\phi = 0 \quad &\text{on } \partial\Omega \end{aligned}\right\} \tag{1.9}$$

admits a solution ϕ with $\phi > 0$ in $\bar{\Omega}$. (If $\alpha'(1) > 0$, then $\lambda_0 > 0$). Indeed, the continuum of equilibria (with values in $(0, 1)$ on Ω) that emerges from $\{(\lambda, 1) : \lambda \in \mathbb{R}\}$ at $\lambda = \lambda_0$ satisfies the alternatives of the global bifurcation theorem of Rabinowitz relative to the ray of equilibria $\{(\lambda, 1) : \lambda \in \mathbb{R}\}$.

In [5], when $\alpha(1) = 1$, we use $\mathcal{C}_0$ and $\mathcal{C}_1$ to denote the continua of equilibria to (1.1) with values in $[0, 1]$ on $\bar{\Omega}$ emanating from $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$ at $\lambda = \lambda^1_{\alpha(0)}(\Omega)$ and from $\{(\lambda, 1) : \lambda \in \mathbb{R}\}$ at $\lambda = \lambda_0$, respectively. We show that both $\mathcal{C}_0$ and $\mathcal{C}_1$ are unbounded in $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$ with the set $\{\lambda \in \mathbb{R} : (\lambda, u) \in \mathcal{C}_i$ for some *non-trivial* u with $0 \leqslant u(x) \leqslant 1$ on $\bar{\Omega}\}$ unbounded in $\mathbb{R}$, for $i = 0, 1$.

Note that solutions to (1.1) along the ray of equilibria $\{(\lambda, 1) : \lambda \in \mathbb{R}\}$ are 'non-trivial' solutions relative to the ray of solutions $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$, and vice versa. Consequently, one way in which $\mathcal{C}_0$ could conceivably be such that the set $\{\lambda \in \mathbb{R} : (\lambda, u) \in \mathcal{C}_0$ for some non-trivial u with $0 \leqslant u(x) \leqslant 1$ on $\bar{\Omega}\}$ is unbounded in $\mathbb{R}$ would be if $\mathcal{C}_0$ links up to $\{(\lambda, 1) : \lambda \in \mathbb{R}\}$. Since $(\lambda_0, 1)$ is the unique point along the ray $\{(\lambda, 1) : \lambda \in \mathbb{R}\}$ at which such a linkage could occur, in such a case $\mathcal{C}_0$ and $\mathcal{C}_1$ would have to coincide. When $\alpha(0) = 0$, equilibria emanating from $\{(\lambda, 1) : \lambda \in \mathbb{R}\}$ at $\lambda = \lambda_0$ necessarily satisfy (1.8). So if $\alpha(0) = 0$, $\alpha'(0) > 0$ and $\alpha'(1) > 0$, the preceding scenario is excluded provided that Ω is not a ball in $\mathbb{R}^N$. In this case, $\mathcal{C}_0$ and $\mathcal{C}_1$ do not intersect. In particular, it follows that, for all sufficiently large values of λ, (1.1) admits at least three equilibria (including $u \equiv 1$) with values in $(0, 1]$ on Ω (see [5] for further detail).

When $\alpha(0) > 0$, the boundary condition in (1.1) does not factor as in (1.6). It follows that one may assume that $\alpha(\mathbb{R}) \subseteq (\delta, R)$ for some fixed $R > \delta > 0$. The results of Amann [3] may again be invoked to guarantee that (1.1) is well posed. It is quite natural to ask whether $\mathcal{C}_0$ and $\mathcal{C}_1$ are distinct (in which case (1.1) admits at least three equilibria with values in $(0, 1]$ on Ω for all sufficiently large values of λ) or if, in fact, $\mathcal{C}_0$ and $\mathcal{C}_1$ coincide, in contrast to the case when $\alpha(0) = 0$ and Ω is not a ball in $\mathbb{R}^N$. We began an examination of this question in [5], but were only able at the time to give a partial answer. We showed that if $\alpha(0) > 0$, then $\mathcal{C}_0$ and $\mathcal{C}_1$ coincide, provided that the function α in the boundary condition of (1.1) is such that

$$\alpha''(u) > 0 \quad \text{and} \quad \alpha'(1) < [\alpha(0)]^3. \tag{1.10}$$

We established the result by showing that if (1.10) held, then the set

$$\Pi_\lambda = \{u \in C^{1,\gamma}(\bar{\Omega}) : (\lambda, u) \text{ is an equilibrium to (1.1) with } 0 < u(x) < 1 \text{ on } \Omega\} \tag{1.11}$$

is empty for all sufficiently large values of λ.

In this paper, we are now able to establish that the conditions we posed in [5] via (1.10) to have $\mathcal{C}_0$ and $\mathcal{C}_1$ coincide are extraneous. Indeed, all that is required to have $\mathcal{C}_0$ and $\mathcal{C}_1$ coincide when $\alpha(0) > 0$ and $\alpha(1) = 1$ is that α be smooth. Explicitly, we require that $\alpha'(1)$ is finite. We present a proof of this result in the next section.

In § 3, we consider what happens if the requirement that $\alpha'(1)$ is finite is lifted. Here we show that if $\alpha'(1) = +\infty$ but α is still Hölder continuous at $u = 1$, then $\mathcal{C}_1$ per se no longer exists (as bifurcation from 1 is no longer possible), while $\mathcal{C}_0$ extends to $+\infty$ in the λ parameter. Since λ_0 in (1.9) tends to $+\infty$ as $\alpha'(1) \to +\infty$, this result is naturally viewed as a limiting case of the result of § 2.

2. The case $\alpha'(1) < \infty$

We first establish the following estimate.

THEOREM 2.1. *Suppose that the function $\alpha(u)$ in (1.1) is continuously differentiable on $\mathbb{R}$ with $\alpha(0) > 0$, $\alpha(1) = 1$, and $\alpha'(1) > 0$. If (λ, u) is a solution of (1.4) with $\lambda > 0$ and $u(x) \in (0,1)$ for $x \in \Omega$, then either*

(i) $\lambda \leqslant \lambda^1_{\alpha(0)}(\Omega) + 1$ *or*

(ii) $$\lambda^1_{\alpha(0)}(\Omega) + 1 < \lambda < \frac{\max_{\bar{\Omega}}((1/h) + 2|\nabla h|^2/h^2)}{\min_{\bar{\Omega}} u_{\alpha(0)}(\lambda^1_{\alpha(0)}(\Omega) + 1)},$$

where $u_{\alpha(0)}(\lambda^1_{\alpha(0)}(\Omega) + 1) > 0$ on $\bar{\Omega}$ is the unique positive solution of

$$\left.\begin{aligned} \nabla^2 u + (\lambda^1_{\alpha(0)}(\Omega) + 1)u(1-u) &= 0 \quad \text{in } \Omega,\\ \alpha(0)\nabla u \cdot \boldsymbol{\eta} + (1-\alpha(0))u &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{2.1}$$

and $h > 0$ on $\bar{\Omega}$ is in $C^{2,\gamma}(\bar{\Omega})$ and satisfies

$$\left.\begin{aligned} \nabla^2 h + 1 &= 0 \quad \text{in } \Omega,\\ \nabla h \cdot \boldsymbol{\eta} + Kh &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.2}$$

Here the constant K is chosen so that

$$K > \sup_{u\in[0,1]} \frac{1-\alpha(u)}{1-u} \cdot \frac{u}{\alpha(u)}. \tag{2.3}$$

Proof. Suppose that u is as in the hypotheses of the theorem and let $w = 1 - u$. Then w satisfies

$$\left.\begin{aligned} -\nabla^2 w + \lambda w u &= 0 \quad \text{in } \Omega,\\ \alpha(u)\nabla w \cdot \boldsymbol{\eta} - \left(\frac{1-\alpha(u)}{1-u}\right) w u &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.4}$$

As in [5], we let $z = wh$, where h satisfies (2.2) and (2.3). It follows from (2.4) that z satisfies

$$\left.\begin{aligned} -\nabla \cdot \left(\frac{1}{h^2}\nabla z\right) - \frac{z}{h}\nabla^2\left(\frac{1}{h}\right) + \lambda \frac{z}{h^2} u &= 0 \quad \text{in } \Omega,\\ \nabla z \cdot \boldsymbol{\eta} + \left(K - \frac{1-\alpha(u)}{1-u}\frac{u}{\alpha(u)}\right) z &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.5}$$

Multiplying the first equation in (2.5) by z, integrating by parts and employing (2.2) and (2.3), we obtain that

$$\begin{aligned}
\lambda \int_\Omega \frac{z^2}{h^2} u \, dx
&= \int_\Omega z \nabla \cdot \left(\frac{1}{h^2} \nabla z \right) dx + \int_\Omega \frac{z^2}{h^2} \nabla^2 \left(\frac{1}{h} \right) dx \\
&= -\int_\Omega \frac{1}{h^2} |\nabla z|^2 \, dx + \int_\Omega \operatorname{div} \left(\frac{z}{h^2} \nabla z \right) dx + \int_\Omega \frac{z^2}{h^2} \nabla^2 \left(\frac{1}{h} \right) dx \\
&= -\int_\Omega \frac{1}{h^2} |\nabla z|^2 \, dx - \int_{\partial\Omega} \left(K - \frac{1-\alpha(u)}{1-u} \frac{u}{\alpha(u)} \right) \frac{z}{h^2} \, dS + \int_\Omega \frac{z^2}{h^2} \nabla^2 \left(\frac{1}{h} \right) dx \\
&\leqslant \int_\Omega \frac{z^2}{h^2} \left(\frac{1}{h^2} + \frac{2|\nabla h|^2}{h^3} \right) dx. \qquad (2.6)
\end{aligned}$$

Since $\alpha(u) \geqslant \alpha(0)$ for all $x \in \partial\Omega$, it follows as in [5] that u is an upper solution to (2.1) provided that $\lambda > \lambda^1_{\alpha(0)}(\Omega) + 1$. Since $\varepsilon\phi$ is a lower solution to (2.1) for $\varepsilon > 0$ sufficiently small, where $\phi > 0$ satisfies (1.5), we have that

$$u \geqslant u_{\alpha(0)}(\lambda^1_{\alpha(0)}(\Omega) + 1) \quad \text{on } \bar{\Omega}, \qquad (2.7)$$

provided that $\lambda > \lambda^1_{\alpha(0)}(\Omega) + 1$. The desired result is now a consequence of (2.6) and (2.7). □

By theorem 2.1, Π_λ in (1.11) is empty for all sufficiently large values of λ. Thus, the discussion in § 1 shows that the only way in which $\mathcal{C}_0$ can be unbounded in $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$ is if it contains the ray $\{(\lambda, 1) : \lambda \in \mathbb{R}\}$ and that the only way that $\mathcal{C}_1$ can be unbounded in $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$ is if it contains the ray of trivial solutions. So theorem 2.1 has as an immediate consequence the following result.

THEOREM 2.2. *Suppose that the function $\alpha(u)$ in (1.1) satisfies the hypotheses of theorem 2.1. Then if $\mathcal{C}_0 \subseteq \mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$ is the continuum of equilibrium solutions (λ, u) to (1.1) with $0 \leqslant u \leqslant 1$ on $\bar{\Omega}$ which emanates from $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$ at $\lambda = \lambda^1_{\alpha(0)}(\Omega)$, where $\lambda^1_{\alpha(0)}(\Omega)$ is as in (1.5), and $\mathcal{C}_1 \subseteq \mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$ is the continuum of equilibrium solutions (λ, u) to (1.1) with $0 \leqslant u \leqslant 1$ on $\bar{\Omega}$ which emanates from $\{(\lambda, 1) : \lambda \in \mathbb{R}\}$ at $\lambda = \lambda_0$, where λ_0 is as in (1.9),*

$$\mathcal{C}_0 \cup (\mathbb{R} \times \{0\}) = \mathcal{C}_1 \cup (\mathbb{R} \times \{1\}).$$

3. The case when $\alpha'(1) = +\infty$

Suppose that $\alpha(\cdot)$ in (1.1) satisfies the hypotheses of the preceding section. Then the rays of equilibria to (1.1) $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$ and $\{(\lambda, 1) : \lambda \in \mathbb{R}\}$ are connected by a continuum of equilibrium solutions (λ, u) with $0 < u < 1$ in $\bar{\Omega}$ at the points $(\lambda^1_{\alpha(0)}(\Omega), 0)$ and $(\lambda_0, 1)$, where $\lambda^1_{\alpha(0)}(\Omega)$ and λ_0 are the principal eigenvalues for (1.5) and (1.9), respectively. As noted in § 1, λ_0 in (1.9) satisfies $\lim_{\alpha'(1) \nearrow \infty} \lambda_0 = +\infty$ [5, formula (3.21)]. As a result, as $\alpha'(1)$ approaches $+\infty$, the bifurcation diagram in $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$ for the equilibria (λ, u) to (1.1) with u values in $[0, 1]$ described in § 2 appears to approximate in some sense the diagram described in [5] for the case when $\alpha(0) = 0$. Our aim in this section is to examine the global bifurcation structure for the problem when the conditions on $\alpha(u)$ in the preceding section are altered so that $\alpha(u)$ is no longer smooth at $u = 1$. We still require $\alpha(0) > 0$, α increasing on $[0, 1]$ and $\alpha(1) = 1$. But now, instead of having $\alpha \in C^1([0,1])$ we will now have $\alpha \in C^1([0,1)) \cap C^\sigma([0,1])$ for some $\sigma \in (0,1)$ with

$$\lim_{u \to 1^-} \alpha'(u) = +\infty. \tag{3.1}$$

In this case, the rays $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$ and $\{(\lambda, 1) : \lambda \in \mathbb{R}\}$ in $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$ remain as equilibria to (1.1). Clearly, there is still bifurcation of equilibria to (1.1) with values in $(0, 1)$ in $\bar{\Omega}$ from $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$ at $\lambda = \lambda^1_{\alpha(0)}(\Omega)$. However, because $\alpha'(1)$ appears in the boundary condition in (1.9), bifurcation from the ray $\{(\lambda, 1) : \lambda \in \mathbb{R}\}$ can no longer be expected to occur at a finite value of λ. The question becomes one of determining the global disposition of the branch of equilibrium solutions (λ, u) to (1.1) with $0 < u(x) < 1$ on $\bar{\Omega}$ that emerges from $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$ at $\lambda = \lambda^1_{\alpha(0)}(\Omega)$. Our approach here is to show that for λ in a bounded interval $[\lambda_1, \lambda_2]$, there is a constant $C = C([\lambda_1, \lambda_2], \alpha(\cdot), \Omega) < 1$ so that for any equilibrium (λ, u) to (1.1) with $\lambda \in [\lambda_1, \lambda_2]$ and $0 < u(x) < 1$ on $\bar{\Omega}$,

$$\max_{x \in \bar{\Omega}} u(x) < C. \tag{3.2}$$

It follows from (3.2) that for all $\lambda \in [\lambda_1, \lambda_2]$ the values of u lie in a range where $\alpha(u)$ remains smooth. As a result, standard regularity theory [7] results may be employed to assure the boundedness of such solutions in $C^{2,\gamma}(\bar{\Omega})$. As a result, continuation principles (e.g. [2, corollary 17.4]) imply that the continuum must extend to $+\infty$ in the λ parameter.

We begin our discussion by showing that equilibrium solutions (λ, u) to (1.1) with $0 < u < 1$ in $\bar{\Omega}$ do not approach the ray $\{(\lambda, 1) : \lambda \geqslant 0\}$ in $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$. Hence, there is no bifurcation of such equilibria from $\{(\lambda, 1) : \lambda \geqslant 0\}$.

PROPOSITION 3.1. *Suppose that $\alpha(u)$ in (1.1) satisfies $\alpha \in C^1([0,1)) \cap C([0,1])$, $\alpha(0) > 0$, α is increasing, $\alpha(1) = 1$ and $\lim_{u \to 1^-} \alpha'(u) = +\infty$. Then the ray $\{(\lambda, 1) : \lambda \geqslant 0\}$ of equilibrium solutions to (1.1) is isolated in $\mathbb{R}^+ \times C^{1,\gamma}(\bar{\Omega})$ from equilibrium solutions (λ, u) to (1.1) with $0 < u < 1$ on $\bar{\Omega}$.*

Proof. If the proposition fails to hold, there is some $\bar{\lambda} \geqslant 0$ and equilibrium solutions (λ_n, u_n) to (1.1) so that $0 < u_n < 1$ on $\bar{\Omega}$ and $(\lambda_n, u_n) \to (\bar{\lambda}, 1)$ in $\mathbb{R} \times C^{1,\gamma}(\bar{\Omega})$. In particular, u_n converges uniformly to 1 on $\partial\Omega$. Then we have

$$\left.\begin{aligned} \nabla^2 u_n + \lambda_n u_n(1 - u_n) &= 0 \quad \text{in } \Omega, \\ \alpha(u_n)\nabla u_n \cdot \boldsymbol{\eta} + (1 - \alpha(u_n))u_n &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.3}$$

Let $w_n = 1 - u_n$ in (3.3). Then

$$\left.\begin{aligned} -\nabla^2 w_n + \lambda_n w_n u_n &= 0 \quad \text{in } \Omega, \\ -\alpha(u_n)\nabla w_n \cdot \boldsymbol{\eta} + \left(\frac{1 - \alpha(u_n)}{1 - u_n}\right) u_n \cdot w_n &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.4}$$

From (3.4) we have that

$$-\frac{\nabla^2 w_n}{w_n} + \lambda_n u_n = 0 \quad \text{in } \Omega. \tag{3.5}$$

Using the divergence theorem to integrate (3.5) we have

$$-\int_{\partial\Omega} \frac{1}{w_n} \nabla w_n \cdot \boldsymbol{\eta} \, \mathrm{d}S - \int_{\Omega} \frac{|\nabla w_n|^2}{w_n^2} \, \mathrm{d}x + \lambda_n \int_{\Omega} u_n \, \mathrm{d}x = 0.$$

From the second equation in (3.4),

$$\frac{1}{w_n} \nabla w_n \cdot \boldsymbol{\eta} = \frac{1-\alpha(u_n)}{1-u_n} \frac{u_n}{\alpha(u_n)}.$$

Hence,

$$\begin{aligned} \lambda_n \int_{\Omega} u_n \, \mathrm{d}x &= \int_{\Omega} \frac{|\nabla w_n|^2}{w_n^2} \, \mathrm{d}x + \int_{\partial\Omega} \left(\frac{1-\alpha(u_n)}{1-u_n} \right) \frac{u_n}{\alpha(u_n)} \, \mathrm{d}S \\ &\geqslant \int_{\partial\Omega} \left(\frac{1-\alpha(u_n)}{1-u_n} \right) \frac{u_n}{\alpha(u_n)} \, \mathrm{d}S \to +\infty, \end{aligned}$$

which is a contradiction, since

$$\lambda_n \int_{\Omega} u_n \, \mathrm{d}x \to \bar{\lambda} \int_{\Omega} \mathrm{d}x = \bar{\lambda}|\Omega|.$$

□

In order to establish the main result of this section, we need the following *a priori* estimate, which is a variant of results in [7] and whose proof is sketched in the appendix.

PROPOSITION 3.2. *Suppose that u satisfies*

$$\begin{aligned} \nabla^2 u &= f(x) \quad \text{in } \Omega, \\ \nabla u \cdot \boldsymbol{\eta} &= g(x) \quad \text{on } \partial\Omega, \end{aligned}$$

where $f(x), g(x) \in C^{\gamma}(\bar{\Omega})$. Then the $C^{1,\gamma}(\bar{\Omega})$ norm of u satisfies

$$\|u\|_{C^{1,\gamma}(\bar{\Omega})} \leqslant C_0(\|g\|_{C^{\gamma}(\bar{\Omega})} + \|f\|_{C^{\gamma}(\bar{\Omega})} + \|u\|_{C(\bar{\Omega})})$$

where C_0 is a constant depending only on γ and Ω.

PROPOSITION 3.3. *Suppose that (λ, u) is an equilibrium solution to (1.1) with $0 < u < 1$ on $\bar{\Omega}$. Suppose that $\alpha \in C^{\sigma}([0,1])$, $0 < \sigma < 1$, with $\alpha(0) > 0$, α increasing and $\alpha(1) = 1$. Then for $\gamma \in (0, \sigma)$ there is a constant C^* depending only on $\|\alpha\|_{C^{\sigma}([0,1])}$, λ and Ω so that $\|u\|_{C^{1,\gamma}(\bar{\Omega})} \leqslant C^*$.*

Proof. We have that

$$\begin{aligned} \nabla^2 u &= f_{\lambda}(u) = \lambda u(u-1) && \text{in } \Omega, \\ \nabla u \cdot \boldsymbol{\eta} &= g(u) = \left(1 - \frac{1}{\alpha(u)}\right) u = h(u)u && \text{on } \partial\Omega. \end{aligned}$$

Since $\alpha(0) > 0$, the function $h(u) = 1 - (1/\alpha(u)) \in C^{\sigma}([0,1])$.

Our intention now is to employ proposition 3.2 to estimate $\|u\|_{C^{1,\gamma}(\bar\Omega)}$. First observe that

$$\|f_\lambda(u)\|_{C^\gamma(\bar\Omega)} = \|\lambda u(1-u)\|_{C^\gamma(\bar\Omega)} \leqslant C\|u\|_{C^\gamma(\bar\Omega)}, \tag{3.6}$$

where C_1 depends only on λ.

We must also estimate $\|g(u)\|_{C^\gamma(\bar\Omega)} = \|h(u)u\|_{C^\gamma(\bar\Omega)}$. Here we begin with $h(u) = h(u(x))$. We have

$$\|h(u)\|_{C^\gamma(\bar\Omega)} = \sup_{\bar\Omega} |h(u)| + [h(u)]_{\gamma;\bar\Omega},$$

where

$$[h(u)]_{\gamma;\bar\Omega} = [h(u(x))]_{\gamma;\bar\Omega} = \sup_{\substack{x_1,x_2\in\bar\Omega \\ x_1\neq x_2}} \frac{|h(u(x_1)) - h(u(x_2))|}{|x_1-x_2|^\gamma}.$$

Observe that since $\alpha(0) > 0$, $\sup_{\bar\Omega}|h(u(x))| \leqslant C$, where C depends on $\bar\Omega$ and $\alpha(\cdot)$. For $[h(u)]_{\gamma;\bar\Omega}$, for $x_1, x_2 \in \bar\Omega$ with $x_1 \neq x_2$ and $u(x_1) \neq u(x_2)$ we have

$$\frac{|h(u)(x_1) - h(u)(x_2)|}{|x_1-x_2|^\gamma} = \frac{|h(u)(x_1) - h(u)(x_2)|}{|u(x_1)-u(x_2)|^\sigma}\left(\frac{|u(x_1)-u(x_2)|}{|x_1-x_2|^{\gamma/\sigma}}\right)^\sigma.$$

It follows that $[h(u)]_{\gamma;\bar\Omega} \leqslant [h]_{\sigma;[0,1]}([u]_{\gamma/\sigma;\bar\Omega})^\sigma$. We know in this case that $\|u\|_{C(\bar\Omega)} \leqslant 1$. Combining these observations, we obtain

$$\begin{aligned}\|h(u)u\|_{C^\gamma(\bar\Omega)} &\leqslant \|h(u)\|_{C(\bar\Omega)} + [h]_{\sigma;[0,1]}([u]_{\gamma/\sigma;\bar\Omega})^\sigma + \|h(u)\|_{C(\bar\Omega)}[u]_{\gamma;\bar\Omega} \\ &\leqslant \tilde C(\|u\|_{C^\gamma(\bar\Omega)} + [\|u\|_{C^{\gamma/\sigma}(\bar\Omega)}]^\sigma + 1),\end{aligned} \tag{3.7}$$

where $\tilde C$ depends on $\bar\Omega$ and $\|h\|_{C^\sigma([0,1])}$.

We may now employ proposition 3.2 in conjunction with (3.6) and (3.7) to obtain

$$\begin{aligned}\|u\|_{C^{1,\gamma}(\bar\Omega)} &\leqslant C_0(\|h(u)u\|_{C^\gamma(\bar\Omega)} + \|f_\lambda(u)\|_{C^\gamma(\bar\Omega)} + \|u\|_{C(\bar\Omega)}) \\ &\leqslant \bar C(\|u\|_{C^\gamma(\bar\Omega)} + (\|u\|_{C^{\gamma/\sigma}(\bar\Omega)})^\sigma + 1),\end{aligned} \tag{3.8}$$

where $\bar C$ depends on $[\lambda_1, \lambda_2]$, $\|h\|_{C^\sigma(\bar\Omega)}$ and $\bar\Omega$. There are two cases in (3.8). If $\|u\|_{C^{\gamma/\sigma}(\bar\Omega)} \leqslant 1$, (3.8) reduces to

$$\|u\|_{C^{1,\gamma}(\bar\Omega)} \leqslant \bar C(\|u\|_{C^\gamma(\bar\Omega)} + 2), \tag{3.9}$$

while if $\|u\|_{C^{\gamma/\sigma}(\bar\Omega)} > 1$, it reduces to

$$\|u\|_{C^{1,\gamma}(\bar\Omega)} \leqslant \bar C(\|u\|_{C^\gamma(\bar\Omega)} + \|u\|_{C^{\gamma/\sigma}(\bar\Omega)} + 1), \tag{3.10}$$

Now, given an $\varepsilon > 0$, [7, lemma 6.35] implies there is a $K = K(\varepsilon, \Omega)$ so that for $\beta = \gamma$ or γ/σ

$$\begin{aligned}\|u\|_{C^\beta(\bar\Omega)} &\leqslant \varepsilon\|u\|_{C^{1,\gamma}(\bar\Omega)} + K\|u\|_{C(\bar\Omega)} \\ &\leqslant \varepsilon\|u\|_{C^{1,\gamma}(\bar\Omega)} + K,\end{aligned} \tag{3.11}$$

since $\|u\|_{C(\bar\Omega)} \leqslant 1$. Combining (3.9) and (3.11) yields

$$\|u\|_{C^{1,\gamma}(\bar\Omega)} \leqslant \bar C\varepsilon\|u\|_{C^{1,\gamma}(\bar\Omega)} + 2\bar C + \bar C K \tag{3.12}$$

and combining (3.10) and (3.12) yields

$$\|u\|_{C^{1,\gamma}(\bar{\Omega})} \leqslant 2\bar{C}\varepsilon\|u\|_{C^{1,\gamma}(\bar{\Omega})} + \bar{C} + 2\bar{C}K. \tag{3.13}$$

Proposition 3.3 now follows from (3.12) and (3.13) by choosing ε so that $2\bar{C}\varepsilon = \frac{1}{2}$. □

We may now establish the main result of this section. Our assertions about the global disposition of the continuum of equilibrium solutions (λ, u) to (1.1) with $0 < u < 1$ on $\bar{\Omega}$ that emanates from $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$ at $\lambda = \lambda^1_{\alpha(0)}(\Omega)$ follow immediately.

THEOREM 3.4. *Suppose that, in addition to the assumptions of proposition 3.3, the function $\alpha(\cdot)$ in (1.1) satisfies (3.1). Then if $\mathcal{A}[\lambda_1, \lambda_2]$ denotes the set*

$$\{u : (\lambda, u) \text{ is an equilibrium solution to (1.1) with } 0 < u < 1 \text{ on } \bar{\Omega},\ \lambda \in [\lambda_1, \lambda_2]\},$$

there is a constant $C = C([\lambda_1, \lambda_2], \alpha(\cdot), \bar{\Omega}) < 1$ so that $\sup_{\bar{\Omega}} u < C$ for any $u \in \mathcal{A}[\lambda_1, \lambda_2]$.

Proof. As in the proof of proposition 3.1, we have that

$$\lambda \int_{\Omega} u \, \mathrm{d}x = \int_{\Omega} \frac{|\nabla w|^2}{w^2} \, \mathrm{d}x + \int_{\partial\Omega} \frac{1 - \alpha(u)}{1 - u} \frac{u}{\alpha(u)} \, \mathrm{d}S,$$

where $w = 1 - u$. Since $0 < u < 1$ on $\bar{\Omega}$,

$$\lambda|\Omega| \geqslant \int_{\partial\Omega} \frac{1 - \alpha(u)}{1 - u} \frac{u}{\alpha(u)} \, \mathrm{d}S. \tag{3.14}$$

Now let $\rho(u) = u/\alpha(u)$. Then $\rho \in C^{\sigma}([0, 1])$ and, for $u \in [0, 1), \rho'(u) = (\alpha(u) - u\alpha'(u))/[\alpha(u)]^2$. By (3.1) there is a value $u^* \in (0, 1)$ so that $\rho'(u) < 0$ for $u^* < u < 1$. Since $\alpha(1) = 1$, it follows that

$$\frac{u}{\alpha(u)} \geqslant 1 \quad \text{for } u^* \leqslant u \leqslant 1. \tag{3.15}$$

Since $\lim_{u\to 1^-} \alpha'(u) = +\infty$, given $K > 0$, there is a $\delta = \delta(K) > 0$ so that if $u > 1 - \delta$, $(1 - \alpha(u))/(1 - u) \geqslant K$. (Without loss of generality we may assume that $\delta < 1 - u^*$.)

We next claim that if $\delta > 0$ is sufficiently small, $\min_{\bar{\Omega}} u < 1 - \delta$ for any $u \in \mathcal{A}[\lambda_1, \lambda_2]$. Suppose otherwise. Let $K > \lambda_2|\Omega|/|\partial\Omega|$ be given. Then, for any $\delta > 0$ with $\delta < \delta(K)$, there is a function $u \in \mathcal{A}[\lambda_1, \lambda_2]$ so that $\min_{\bar{\Omega}} u \geqslant 1 - \delta$. Since u satisfies

$$\nabla^2 u + \lambda u(1 - u) = 0$$

in Ω, $\min_{\bar{\Omega}} u$ is achieved on $\partial\Omega$. In particular, $\min_{\partial\Omega} u \geqslant 1 - \delta > 1 - \delta(K)$. So, for any $x \in \partial\Omega$, $u(x) > 1 - \delta(K) > u^*$, where u^* is as in (3.15). It follows that

$$\int_{\partial\Omega} \frac{1 - \alpha(u)}{1 - u} \cdot \frac{u}{\alpha(u)} \, \mathrm{d}s \geqslant K|\partial\Omega|.$$

Hence, it follows from (3.14) that $K \leqslant \lambda_2|\Omega|/|\partial\Omega|$, which is a contradiction.

So we now have for any sufficiently small $\delta > 0$ that $\min_{\bar{\Omega}} u < 1 - \delta$ for any $u \in \mathcal{A}[\lambda_1, \lambda_2]$. Proposition 3.3 implies there is a $C^* = C^*([\lambda_1, \lambda_2], \alpha(\cdot), \Omega)$ so that $\|u\|_{C^{1,\gamma}(\bar{\Omega})} < C^*$ for any $u \in \mathcal{A}[\lambda_1, \lambda_2]$. In particular, $\|u\|_{C^1(\bar{\Omega})} < C^*$. Now let $u \in \mathcal{A}[\lambda_1, \lambda_2]$ correspond to $\lambda \in [\lambda_1, \lambda_2]$. Let $\bar{x} \in \partial\Omega$ so that $u(\bar{x}) = \min_{\bar{\Omega}} u$. Since $\|u\|_{C^1(\Omega)} < C^*$, it follows that

$$u(x) - u(\bar{x}) \leqslant C^*|x - \bar{x}|$$

for $x \in \bar{\Omega}$. Hence,

$$\begin{aligned} u(x) &\leqslant C^*|x - \bar{x}| + u(\bar{x}) \\ &\leqslant C^*|x - \bar{x}| + 1 - \delta \\ &< 1 - \tfrac{2}{3}\delta \end{aligned}$$

provided $|x - \bar{x}| \leqslant \delta/3C^*$. Let $r = \delta/3C^*$. We have that if $x \in \bar{\Omega} \cap B_r(\bar{x})$, $u(x) < 1 - \frac{2}{3}\delta$.

Suppose that the theorem is false. Then there is a sequence (λ_n, u_n) of equilibrium solutions to (1.1) with $\lambda_n \in [\lambda_1, \lambda_2]$, $0 < u_n < 1$ on $\bar{\Omega}$ such that $\sup_{\bar{\Omega}} u_n \to 1$ as $n \to \infty$. Let $\bar{x}_n$ be the point on $\partial\Omega$ associated with u_n via $u_n(\bar{x}_n) = \min_{\bar{\Omega}} u_n$. Without loss of generality, we may assume $\bar{x}_n \to x^* \in \partial\Omega$. For all sufficiently large values of n, $\bar{x}_n \in B(x^*, \frac{1}{2}r)$ so that

$$u_n(x) < 1 - \tfrac{2}{3}\delta \quad \text{for } x \in \partial\Omega \cap B(x^*, \tfrac{1}{2}r). \tag{3.16}$$

Now choose $f \in C^{2,\gamma}(\bar{\Omega})$ so that

$$\alpha(1 - \tfrac{2}{3}\delta) < f(x) < \alpha(1 - \tfrac{1}{3}\delta) \quad \text{on } B(x^* \tfrac{1}{4}r), \tag{3.17}$$

$$f(x) \equiv 1 \quad \text{on } \bar{\Omega} \setminus B(x^*, \tfrac{1}{2}r), \tag{3.18}$$

$$\alpha(1 - \tfrac{2}{3}\delta) \leqslant f(x) \leqslant 1 \quad \text{on } \bar{\Omega}, \tag{3.19}$$

and consider the problem

$$\left.\begin{aligned} \nabla^2 v + \lambda_2 v(1 - v) &= 0 \quad \text{in } \Omega, \\ f(x)\nabla v \cdot \boldsymbol{\eta} + (1 - f(x))v &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.20}$$

It follows from (3.16)–(3.19) that, for all sufficiently large values of n, we have $\alpha(u_n(x)) \leqslant f(x)$ for all $x \in \partial\Omega$. It now follows as in [5, 6] that $u_n(x)$ is a lower solution to (3.20) for all such n. By (3.17)–(3.19), 1 is a strict upper solution to (3.20). So, for any sufficiently large n, the method of upper and lower solutions guarantees the existence of a solution v to (3.20) with $u_n(x) < v(x) < 1$ on $\bar{\Omega}$. Since such a v is necessarily unique [4], $\sup_{\bar{\Omega}} u_n < \sup_{\bar{\Omega}} v < 1$ for all sufficiently large n, which is a contradiction. Thus, the theorem must hold. □

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Appendix A. Sketch of the proof of proposition 3.2

Proposition 3.2 is an analogue of [7, theorem 8.33]. In the former result, $C^{1,\gamma}(\bar{\Omega})$ estimates on solutions to generalized Dirichlet boundary-value problems are given in terms of the $C(\bar{\Omega})$ norm of the solution, the $C^{\gamma}(\bar{\Omega})$ norm of the internal data f and the $C^{1,\gamma}(\bar{\Omega})$ norm of the boundary data g. In proposition 3.2 the boundary operator is now a Neumann operator and [7] does not explicitly state a companion result to theorem 8.33. However, by carefully examining the proof of theorem 8.33 and the way in which estimates for classical Dirichlet boundary problems are modified in [7, § 6.7] to treat other boundary operators, one may obtain proposition 3.2.

Recall that *a priori* estimates of the type in the statement of proposition 3.2 are obtained by piecing together local *a priori* estimates throughout Ω and along $\partial\Omega$. Inside Ω, the coefficients of the operator are approximated by constants in balls of small radius, which allows one to make linear changes of coordinates that reduce the problem locally to a Poisson equation. Along the boundary $\partial\Omega$, an additional initial change of coordinates that 'flattens' the boundary is needed, so that one considers the problem in the intersection of a ball of small radius with a half-space of dimension equal to that of Ω. If non-zero boundary data g is prescribed in the problem, it reduces to a homogeneous problem by finding *a priori* estimates for $v = u - \psi$ instead of u, where the boundary operator applied to ψ yields the given boundary data g. For the case of a Dirichlet boundary operator, ψ is simply the boundary data g. In such a case, g needs to be extended so as to be defined throughout $\bar{\Omega}$ and g must have the level of regularity one desires for the solution u. However, in the case of a Neumann operator, as in [7, § 6.7], ψ is defined locally by what amounts to an appropriately scaled convolution of g against a suitable compactly supported smooth positive test function (see [7, equation (6.68), § 6.7]). In this case, one obtains bounds on the Hölder exponents of the partial derivatives of ψ in terms of the Hölder exponent of g, since $\partial x_i(g * \varphi) = g * \partial_{x_i}\varphi$ for a smooth compactly supported test function φ, where '$*$' denotes convolution. With this accommodation, the estimate in proposition 3.2 follows as in [7, § 8.1].

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